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LLL vs. LLM:

Half BPS Sector of $\mathcal{N} = 4$ SYM $\equiv$ Quantum Hall System

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Abstract

In this paper we elaborate on the correspondence between the quantum Hall system with filling factor equal to one and the $\mathcal{N} = 4$ SYM theory in the 1/2 BPS sector, previously mentioned in the [hep-th/0409174, 0409115]. We show the equivalence of the two in various formulations of the quantum Hall physics. We present an extension of the noncommutative Chern-Simons Matrix theory which contains independent degrees of freedom (fields) for particles and quasiholes. The BPS configurations of our model, which is a model with explicit particle-quasihole symmetry, are in one-to-one correspondence with the 1/2 BPS states in the $\mathcal{N} = 4$ SYM. Within our model we shed light on some less clear aspects of the physics of the $\mathcal{N} = 4$ theory in the 1/2 BPS sector, like the giant dual-giant symmetry, stability of the giant gravitons, and stringy exclusion principle and possible implications of the (fractional) quantum Hall effect for the AdS/CFT correspondence.

1 Introduction

In 1973 't Hooft showed that [1] all correlators of a $U(N)$ gauge theory, including $\mathcal{N}=4$ D=4 Supersymmetric Yang-Mills (SYM) theory, admit a double expansion which can be arranged in powers of Ng_{YM}^2 and $1/N$ where the $1/N$ expansion is encoding the topology of the corresponding Feynman diagrams, as we have in string theory. This observation found a full realization within the celebrated AdS/CFT framework [2].

In the $\mathcal{N}=4$ $U(N)$ SYM, there are specific sectors, i.e. set of operators, in which the above double expansion reduces to a single expansion in powers of $1/N$. The operators of this sector are chiral primaries (and their descendants) whose scaling dimension is exact and the g_{YM} corrections are absent due to supersymmetry. In fact the large amount of supersymmetry removes any g_{YM}, perturbative or non-perturbative, dependence in all the n-point functions of chiral primaries. The chiral primary operators preserve 16 supersymmetries of the superconformal algebra of the $\mathcal{N}=4$ SYM theory, $PSU(2,2|4)$, and hence they all belong to the 1/2 BPS sector of the theory. As such 1/2 BPS sector provides us with a laboratory to concentrate on the 1/N behavior and the combinatorics of the n-point functions.

In the 1/2 BPS sector, which will be our main focus in this paper, the $\mathcal{N}=4$ $U(N)$ SYM simplifies significantly and essentially becomes equivalent to a system of N 2d fermions [3, 4]. These fermions are living in a specific sector of a 2d harmonic oscillator potential. The 1/2 BPS condition restricts the dynamics of the system further down to a one dimensional N fermion system, with degenerate energy levels. These facts will be reviewed and explained further in section 3.1.

One would try to explore the fermionic nature appearing in the analysis of the 1/2 BPS sector in the dual gravity picture. This has been carried out in a novel work by Lin-Lunin-Maldacena (LLM) [5]. LLM constructed type IIB supergravity solutions compatible with the supersymmetries preserved in the 1/2 BPS sector of $\mathcal{N}=4$ SYM; i.e. LLM found *static, non-singular* deformations of $AdS_5\times S^5$ geometry preserving at least 16 supercharges with $SO(4)\times SO(4)\times U(1)\subset SO(4,2)\times SO(6)$ isometries, the $U(1)$ corresponding to translations along the globally defined time-like (or in special cases light-like) Killing vector. The solutions of LLM are only determined through a single function z, which is a function of three coordinates usually denoted by (x_1,x_2) and a non-negative coordinate y, and z/y^2 satisfy a six dimensional Laplace equation [5]. The LLM geometries are then completely specified by giving the value of the function z at $y=0$. The smoothness condition forces $z_0(x_1,x_2)\equiv z(x_1,x_2,y=0)$ to take only values $+1/2$ or $-1/2$, a very restrictive and strong condition. Hence, LLM made a one-to-one correspondence between the 1/2 BPS sectors of $AdS_5\times S^5$ deformations and the fermionic description of the $\mathcal{N}=4$ SYM, by directly identifying (x_1,x_2) plane with the phase space of the one dimensional fermions mentioned earlier. Explicitly this identification implies that

$$[x_1,x_2]=2\pi i\ l_p^4\ , \tag{1.1}$$

where l_p is the ten dimensional Planck length. That is, the (x_1, x_2) plane in the geometry turns out to be a noncommutative Moyal plane, a result coming from and confirmed by the quantum gravity considerations (resulting via AdS/CFT from the $\mathcal{N} = 4$ SYM description). To be precise and to make the above correspondence exact, LLM borrowed one fact from the semiclassical physics, namely quantization of the supergravity fiveform flux. The smoothness condition $z_0 = \pm 1/2$, is then closely related to the Pauli exclusion principle in the fermion picture. LLM used a convenient color coding: denote the $z_0 = -1/2$ with black and $z_0 = +1/2$ with white. In the fermion picture, $z_0 - 1/2$ is the fermion density (in its phase space), i.e. black regions are filled with fermions and white regions are empty.

The system of 2d fermions moving in a constant background magnetic field has an interesting sector in which energy levels of fermions are degenerate and fermions are labeled by their angular momentum quantum number, the lowest Landau level (LLL). In the LLL the phase space of fermions is equivalent to the one we encountered in the 1/2 BPS sector of $\mathcal{N} = 4$ SYM. On the other hand (almost) all the interesting physics of the Quantum Hall Effect (QHE) can be described in terms of LLL's. Therefore, one should be able to build a dictionary between $\mathcal{N} = 4$ SYM 1/2 BPS operators (or LLM geometries) and the quantum Hall physics, within which one can learn more about the $\mathcal{N} = 4$ SYM and hence quantum gravity from QHE. This is indeed what we are aiming for in this paper. Some preliminary steps in this direction have been taken in [6].

To build the QHE/SYM correspondence we need to review, very briefly, the standard descriptions of QHE. This is done in section 2, where we present three different formulations for studying quantum Hall systems. In section 3, we show how the 1/2 BPS sector of SYM is directly mapped into either of the quantum Hall descriptions. We show this at the level of the actions and Hilbert spaces. As has also been mentioned in [6], we will see $\mathcal{N} = 4$ SYM is a specific quantum Hall system with filling factor $\nu = 1$. We then focus on the quasiparticles of the corresponding quantum Hall system and argue that they are mapped to giant gravitons [7, 8] in the gravity (gauge theory) picture.

Quantum Hall system with $\nu = 1$ has a particle/quasihole symmetry. In section 4 we elaborate further on this issue. There we present an extension of the Chern-Simons matrix model action by promoting the quasiholes to new degrees of freedom, new fields. That is we present an effective field theory consisting of two fields, one for particles and one for quasiholes. The BPS solitons of our proposed action exhibit particle/quasihole symmetry. In section 4 we present several solutions of these BPS equations and show that our BPS configurations are in one-to-one correspondence with chiral primary operators of $\mathcal{N} = 4$ SYM. Using our proposed action we study stability of its BPS solitons and show that classically there is no transition between giant graviton states. The last section is devoted to discussions on our results and further extensions and generalizations of QHE/SYM correspondence.

2 Lightening Review of the Quantum Hall System

Integer Quantum Hall effect (IQHE) is simply quantization of the Hall conductance in e^2/h units as $\sigma_{xy} = \nu e^2/h$, where ν is an integer quantum number. This phenomenon is exhibited by condensed matter systems which can be approximated as an effectively two dimensional ideal electron gas living in a strong magnetic field. Fractional values of the quantum number ν has also been observed. The corresponding Hall effect is called Fractional Quantum Hall effect (FQHE). Fractional and integer quantum hall effects have two quite different underlying physics. Physics of the FQHE involves strong correlations among the electrons. Collective excitations of the electron gas in the fractional case have fractional charges and statistics which is something between ordinary Bose and Fermi statistics. Quantum description of the Hall system for $\nu^{-1} \in \mathbb{Z}$ is given by the so-called Laughlin wavefunction. Laughlin wave function encodes the edge fluctuations of an incompressible gapless fluid. There also exists an algebraic approach to the Hall problem. The algebra of the area preserving diffeomorphisms in two dimensions (w_∞ algebra) has been studied before e.g. see [17]. The quantum version of this algebra so called $W_{1+\infty}$ describes the edge excitations of the Hall droplet. For more details refer to [15, 16].

There have been three different approaches to quantum Hall system in the literature. The more standard one is based on quantum mechanics of some non-relativistic fermions in an (strong) external magnetic field in the Lowest Landau Level (LLL) (for review see [14]) and second one is the effective field theory description in terms of (noncommutative) Chern-Simons gauge theory [12]. The third one is the Matrix Chern-Simons theory which interpolates between the field theory description and the quantum mechanical one. In this section we briefly review these three approaches.

2.1 Landau problem and Quantum Hall effect

Here we study the Landau problem in two different bases. This would enable us to draw an explicit connection between the Landau problem and the half BPS sector of the $\mathcal{N}=4$ SYM.

Let us start with a single non-relativistic charged particle moving in a two dimensional plane transverse to a constant magnetic field i.e., the Landau problem. The Hamiltonian for the system is

$$H = \frac{1}{2m}(p_i - \frac{eB}{2c}\epsilon_{ij}x_j)^2 \ , \tag{2.1}$$

where $i, j = 1, 2$ and x_i, p_i are the corresponding phase space conjugate variables and B is the strength of the magnetic field. Next let $\Pi_i = p_i - \frac{eB}{2c}\epsilon_{ij}x_j$, it is then readily seen that

$$[\Pi_1, \Pi_2] = -i\hbar\frac{eB}{c} \ , \tag{2.2}$$

and hence if we call $Y_1 = \frac{c}{eB}\Pi_2$, $[Y_1, \Pi_1] = i\hbar$, the Hamiltonian takes the form

$$H = \frac{1}{2m}\left(\Pi_1^2 + (\frac{eB}{c})^2 Y_1^2\right) \ , \tag{2.3}$$

which is the Hamiltonian for a simple one dimensional Harmonic oscillator with frequency

$$\omega_0 = \frac{eB}{mc} \ . \tag{2.4}$$

The spectrum of the Hamiltonian is

$$E = \hbar\omega_0(n + \frac{1}{2}) \ . \tag{2.5}$$

Noting that $\mathcal{J} = \epsilon_{ij}x_i p_j$ also commutes with the Hamiltonian the energy eigenstates are infinitely degenerate and their degeneracy is labeled by J, eigenvalues of $\mathcal{J}$, which can be any arbitrary integer.

Now, let us analyze the above Hamiltonian in another way. Expand the square to obtain

$$H \ = \ \frac{1}{2m}p^2 + \frac{e^2B^2}{8mc^2}x^2 + \frac{eB}{2mc}\epsilon_{ij}x_i p_j \ . \tag{2.6}$$

The first two terms of the above Hamiltonian is the Hamiltonian for a two dimensional harmonic oscillator, H_0, with frequency $\omega_0/2$. The spectrum of this part is

$$E_0 = \frac{1}{2}\hbar\omega_0(n + n' + 1) \ . \tag{2.7}$$

The next part is proportional to the angular momentum which has the spectrum $J = \hbar(n' - n)$, where both n and n' are non-negative integers. Putting these two contributions together we obtain the spectrum of the whole Hamiltonian H to be exactly given by (2.5).

The lowest energy state, the lowest Landau level (LLL), is then given by $n = 0$ but arbitrary n'. In the lowest Landau level, ignoring the zero point energy, H_0 is proportional to J, i.e.

$$H_0 - \frac{1}{2}\hbar\omega_0 = \frac{1}{2}\hbar\omega_0 J \ . \tag{2.8}$$

In the lowest Landau level, which is known to describe the quantum Hall physics, the Hamiltonian is essentially J, or the action corresponding to the system is

$$S = \frac{eB}{2c}\int dt\epsilon_{ij}x_i\dot{x}_j \ . \tag{2.9}$$

2.2 Fluid description of Quantum Hall effect

Consider a system of finite number of non-relativistic interacting particles. A continuum (field theory) description of this fluid is obtained by promoting the particle labels to a continuous co-moving coordinate y. The real space density is given by

$$\rho = |\frac{\partial y}{\partial x}|\rho_0 \ , \tag{2.10}$$

where x is the real space position. It can be seen that if the fluid is incompressible the corresponding continuum Lagrangian has a gauge invariance under area preserving diffeomorphisms (APD) in the y plane [12]. Small fluctuations around the static background solution $x_i = y_i$ is closely connected to Electrodynamics if we parameterize these fluctuations as

$$x_i = y_i + \epsilon_{ij}\frac{A_j}{2\pi\rho_0} \ , \tag{2.11}$$

where A plays the role of electromagnetic vector potential.

Now consider a fluid with charged particles moving in a constant magnetic field. The Lagrangian acquires a new term induced by the background magnetic field. The APD's would still keep the new action invariant and (2.10) can be rewritten as

$$1 = \frac{1}{2}\epsilon_{ij}\epsilon_{ab}\frac{\partial x_b}{\partial y_j}\frac{\partial x_a}{\partial y_i} \equiv \frac{1}{2}\epsilon_{ab}\{x^a, x^b\}_{P.B.} \ . \tag{2.12}$$

In the strong magnetic field limit the action is basically dominated by the magnetic field term (setting $c = 1$), i.e.

$$S = \frac{eB\rho_0}{2}\int dt d^2y \epsilon_{ab} x^a \dot{x}^b \ .$$

(Dropping other terms in the action is equivalent to restricting to lowest Landau level.) In this limit the equation of motion and the constraint can be encapsulated in a single action via introducing a non-dynamical time component of A, A_0:

$$L = \frac{eB\rho_0}{2}\epsilon_{ab}\int d^2y[(\dot{X}_a - \frac{1}{2\pi\rho_0}\{X_a, A_0\})X_b + \frac{\epsilon_{ab}}{2\pi\rho_0}A_0] \ , \tag{2.13}$$

where the bracket is the Poisson bracket defined in (2.12).

This theory admits vortex solutions. Chern-Simons vortex is basically radial disturbance of the fluid toward or away from the center of the vortex proportional to the q/r where q is related to the excess or deficit charge of the vortex by

$$e_{qp} = \rho_0 q e \ , \tag{2.14}$$

and r measures the distance from the center of the vortex. These are quasihole or quasiparticle states of the Hall fluid in the continuum description. Semiclassical quantization of this theory implies that [12]

$$e_{qp} = 2\pi\frac{\rho_0}{B} = \nu e \ , \tag{2.15}$$

where $\nu = \frac{2\pi\rho_0}{eB}$ is the filling fraction. Filling fraction is the ratio of the number of the electrons to the magnetic flux (that is, inverse of magnetic flux per particle). Full quantization of the theory gives rise to quantization of the ν inverse. Filling fraction also controls statistics of the collective excitations of the fluid.

2.3 NC Chern-Simons Matrix model description of QHE

APD's (or w_∞ algebra), which reflect symmetry of the system under relabeling of the particles in an incompressible fluid, translate into the gauge symmetries of the continuum theory. A gauge theory based on this gauge invariance would be able to capture some long distance physics but it is unable of incorporating the intrinsic granular structure of the fluid. It turns out that a description of the system which is more faithful to the underlying discrete physics is a matrix model description of the fluid. In this description classical configuration of the N number of electrons is replaced by the space of $N \times N$ Hermitian matrices. The action (2.13) can be generalized to a matrix theory [12]

$$L = \frac{eB}{2}\epsilon_{ab}Tr(\dot{X}_a - i[X_a, A_0])X_b + eB\theta Tr A_0 \ , \tag{2.16}$$

where $\theta = 1/(2\pi\rho_0)$ plays the role of the noncommutativity parameter. In this action the APD's are replaced by the $U(N)$ gauge symmetry. Constraint equation is obtained by varying this action with respect to A_0

$$[X_a, X_b] = i\theta\epsilon_{ab} \ . \tag{2.17}$$

Although we started with a finite N, due to antisymmetric nature of the commutator,(2.17) can only be solved for infinite size matrices. Therefore, the model describes "infinite" number of particles. We will return to this point momentarily.

It is worth emphasizing that there are two different kinds of noncommutativity not to be confused with each other. One of them is controlled by $1/(eB)$ and has a quantum mechanical origin; X_1 and X_2 are canonically conjugate and hence do not commute as quantum operators, i.e.

$$[\hat{X}_1, \hat{X}_2]_{opt.} = \frac{i}{eB} \ .$$

The second one is encoding the APD invariance of the theory (in the continuum case) or its permutation subgroup (in the discrete case i.e. the matrix model description). The latter noncommutativity is controlled by $\theta = 1/(2\pi\rho_0)$

$$[X_1, X_2]_{Mat.} = i\theta \ .$$

Statistics of the Chern-Simons particles is determined by the filling fraction ν. The ratio of the two noncommutativities, $\nu^{-1} = eB\theta$ is ought to be an integer if we demand the action (2.16) to be invariant under large gauge transformations

[18]. It can be shown that there is a density-statistics connection; depending on $\nu = 1/(2n+1)$ or $\nu = 1/(2n)$ for integer n, the Hall fluid excitations are either Fermions or Bosons.

At this point it is instructive to pause for a moment and make a comparison with the $\mathcal{N}$=4 SYM theory in its half BPS sector. It is rather clear that the above mentioned matrix model cannot have an origin as a particular sector of a "finite" N ($U(N)$ is the gauge group) SYM theory. It is impossible to satisfy the APD invariance constraint by means of finite dimensional matrices as of finite N $\mathcal{N}$=4 SYM theory. As it was explained earlier the same problem occurs if one is to write a matrix model for a Hall system with finite number of particles.

Matrix model for Hall systems with finite number of particles has been discussed in [13]. Finite N matrix models can be constructed by introducing new degrees of freedom so-called edge states. In the presence of the edge state commutator gets modified. This anomaly provides the opportunity to satisfy the APD invariance constraint by finite N matrices. Now we proceed to review some general aspects of the finite N Chern-Simons matrix model.

2.4 Finite dimensional NC Chern-Simons Matrix models

The starting point is to modify the action (2.16) with extra degrees of freedom called edge state [13]

$$L = \frac{B}{2} Tr\epsilon_{ab}(\dot{X}_a + i[A_0, X_a])X_b + B\theta A_0 + B\Psi^\dagger(i\dot{\Psi} - A_0\Psi) - \frac{1}{2}\omega^2(X_a)^2, \quad (2.18)$$

where Ψ, the edge state, is a complex valued vector field in the fundamental of the $U(N)$ gauge group and we have set $e = 1$. The X^2 term has been added to make a droplet the lowest energy state [13] and setting $\omega = 0$ (2.18) reduces to NC Chern-Simons Matrix model. The constraint gets modified as follows

$$-i[X_1, X_2] + \Psi\Psi^\dagger - \theta = 0 \ . \quad (2.19)$$

Taking the trace would not lead to any inconsistency instead it gives

$$\Psi^\dagger\Psi = N\theta \ . \quad (2.20)$$

Using equation of motion for Ψ in temporal gauge, one obtains

$$\Psi = \sqrt{\theta N}|v\rangle \ , \quad (2.21)$$

where $|v\rangle$ is an arbitrary constant unit vector. The constraint (2.19) now reads

$$[A, A^\dagger] = 2\theta(1 - N|v\rangle\langle v|) \ , \quad (2.22)$$

where

$$A = X_1 + iX_2 \ . \quad (2.23)$$

There are various finite dimensional solutions to the above constraint equation corresponding to different Hall states. For instance

$$
\begin{aligned}
A &= \sum_{n=0}^{N-1} \sqrt{2n\theta}|n-1\rangle\langle n| \ , &(2.24)\\
|v\rangle &= |N-1\rangle \ , &(2.25)
\end{aligned}
$$

where (2.25) is only a freedom in gauge choice originating from the time independent part of the $U(N)$ gauge invariance of the theory, is a solution representing a circular quantum Hall droplet of radius $\sqrt{2N\theta}$. $|m\rangle$ is a harmonic oscillator basis. The radius-squared matrix coordinate $R^2 = X_1^2 + X_2^2 = A^\dagger A$ is diagonal in the harmonic basis and its highest eigenvalue goes like $N\theta$ which suggests that we are talking about a finite size quantum Hall state or Hall *droplet* [13]. There are other types of excitations which will be of our interest later on: quasihole states. Quasihole states with charge $-q$ at the origin satisfying (2.19) can be constructed as well

$$
A = \sqrt{2\theta}(\sqrt{q}|N-1\rangle\langle 0| + \sum_{n=1}^{N-1} \sqrt{n+q}|n-1\rangle\langle n|) \ . \qquad (2.26)
$$

Looking at R^2 eigenvalues reveals that the lowest eigenvalue is proportional to $2\theta q$. This means that there is a circular hole of area proportional to $2\theta q$ at the origin. Of course the radius of the droplet itself has also changed to take care of the total number of the particles inside the droplet which is fixed to be N. There are no quasiparticle excitations (accumulation of the particles) in this model.

3 Building the SYM/Quantum Hall Dictionary

In the previous section we reviewed various approaches to quantum Hall problem and made connections between them. In this section we show that how $\mathcal{N} = 4$ $U(N)$ SYM theory, in the 1/2 BPS sector, is related to quantum Hall problem and each of the above approaches. This will be done first at the level of the corresponding actions and then by relating the 1/2 BPS SYM operators and the quantum Hall states. We also briefly discuss the gravity picture via AdS/CFT duality.

3.1 Effective action in the 1/2 BPS sector

Let us start with the $\mathcal{N} = 4$ $U(N)$ SYM action on $R \times S^3$ and denote one of the three complex scalars present in the $\mathcal{N} = 4$ gauge multiplet by Z. In the 1/2 BPS sector the operators can only be made out of Z and moreover these operators cannot have non-trivial dependence on the S^3. In the 1/2 BPS sector we should preserve $SO(4) \times SO(4) \times U(1) \subset SO(4,2) \times SO(6)_R$ of the theory and are only allowed to perturb the theory by chiral primary operators, i.e. operators only made out of Z

and not $Z^\dagger$ or any other fields [3, 4]. Therefore, the action relevant to this sector is simply

$$S_{reduced} = \frac{1}{2}\int dt\; Tr\left((D_0 Z)^\dagger D_0 Z - Z^\dagger Z - \frac{1}{2}[Z, Z^\dagger]^2\right), \tag{3.1}$$

where we have used the conformal invariance of the theory and chosen the radius of the S^3 such that there are no prefactors in the action and we have also rescaled t, Z, A_0 such that they are all dimensionless. The above action, ignoring the last term, is the action for N^2 uncoupled two dimensional harmonic oscillators with frequency one. One should, however, remember that not all the elements of the $N \times N$ matrices are independent and dynamical, as they may be related by the $U(N)$ gauge transformations. We will come back to the issue of gauge fixing later on in this section.

The Dilatation operator in the sector containing operators only made out of Z and $Z^\dagger$, in the first lowest order in g^2_{YM} is of the form:[1]

$$\mathcal{D} = Tr\left(Z\frac{\delta}{\delta Z}\right) + Tr\left(Z^\dagger\frac{\delta}{\delta Z^\dagger}\right)\ . \tag{3.2}$$

In the same sector the R-charge J is measured by

$$\mathcal{J} = Tr\left(Z\frac{\delta}{\delta Z}\right) - Tr\left(Z^\dagger\frac{\delta}{\delta Z^\dagger}\right)\ . \tag{3.3}$$

Note that (3.2) is nothing but the Hamiltonian for N^2 2d harmonic oscillators with the same frequency. In the 1/2 BPS sector $\mathcal{D} - \mathcal{J} = 0$ and hence in this sector

$$\mathcal{D} = Tr\left(Z\frac{\delta}{\delta Z}\right)\ . \tag{3.4}$$

(In this sector we are dealing with the operators which are only made out of Z.) In the 1/2 BPS sector there is no g_{YM} dependence in the scaling dimensions of the operators and hence the dilatation operator (3.2) is exact. As it is manifest in the 1/2 BPS sector we are only left with an effectively one dimensional system out of the 2d harmonic oscillator system we started with.

$\frac{\delta}{\delta Z}$ is the momentum conjugate to Z, Π_Z. On the other hand from the $\mathcal{N} = 4$ SYM action, in the temporal gauge, we have $\Pi_Z = \dot{Z}^\dagger$. Therefore, the action for a system with the Hamiltonian $\mathcal{D} - \mathcal{J}(= 0)$ in the BPS sector is simply

$$S_{1/2\ BPS} = \frac{i}{2}\int dt\; Tr\left(Z^\dagger \dot{Z} - (\dot{Z})^\dagger Z\right)\ . \tag{3.5}$$

It is very instructive to compare the above discussions and formulae with those of sections 2.1 and 2.4. In terms of quantities in section 2.1, it is explicitly seen that

[1] It is worth noting that the D-term $[Z, Z^\dagger]^2$ in the action (3.1) does not contribute to the dilatation operator.

the two are related by

$$\begin{aligned} \mathcal{D} &\leftrightarrow H_0 - \hbar\omega_0/2 \\ \mathcal{J} &\leftrightarrow \mathcal{J} \\ \mathcal{D} - \mathcal{J} &\leftrightarrow H \end{aligned} \tag{3.6}$$

Hence $\mathcal{D} - \mathcal{J}$ is the Hamiltonian of N fermions (see the arguments below) in the external magnetic field and it is seen that going to LLL corresponds to taking the BPS states for which $\mathcal{D} - \mathcal{J}$ vanishes.

The action (3.5), with $N \times N$ matrices, is equivalent to the Polychronakos finite N matrix Chern-Simons theory discussed in section 2.4 after fixing the temporal gauge $A_0 = 0$ and replacing the edge state Ψ with the specific classical solution given in (2.21). For the equivalence, however, one should still impose the constraint (2.19).

So far we have established the relation between the quantum Hall system and the 1/2 BPS sector of $\mathcal{N} = 4$ SYM at classical level, i.e. at the level of the actions. As the next step we would like to push this further to the level of partition function. Explicitly, we want to show that the partition function of the $\mathcal{N} = 4$ $U(N)$ SYM in the 1/2 BPS sector is producing the "Laughlin" wavefunction [14], that is

$$Z_{1/2\ BPS} = \int \left(DZD\bar{Z}|_{1/2\ BPS}\right)\ e^{-S_{1/2\ BPS}} = \langle \psi_L | \psi_L \rangle\ , \tag{3.7}$$

with

$$\psi_L = \prod_{i>j=1}^{N} (z_i - z_j)\ e^{-\sum_{i=1}^{N} \bar{z}_i z_i/2}\ , \tag{3.8}$$

where $z_i, \bar{z}_i$ $i = 1, 2, \cdots, N$ are the eigenvalues of the $N \times N$ matrices of $U(N)$ $\mathcal{N} = 4$ SYM, Z, $\bar{Z}$. To show this we need to use the $U(N)$ gauge symmetry to diagonalize Z. We fix the temporal gauge $A_0 = 0$ and use the remaining time independent (global) gauge transformations to diagonalize Z. This, however, is not possible with a single $U(N)$, as Z is a complex (non-hermitian) $N \times N$ matrix. In order to diagonalize Z we need to double $U(N)$ to $U(N) \times U(N)$ or to complexify $U(N)$ to $U(N, \mathbb{C})$. Since we are going to reduce the computations to 1/2 BPS sector indeed we can use this extended gauge group. To see this it is more convenient to use the Hamiltonian path integral with measure $DZD\bar{Z}D\Pi_Z D\Pi_{\bar{Z}}$. In the 1/2 BPS sector, however, $\Pi_Z = i\bar{Z}$ and $\Pi_{\bar{Z}} = -iZ$. This means that in order to do computations with the path integral in the 1/2 BPS sector in the process of the gauge fixing we need to divide the measure by $Vol_{U(N)} \times Vol_{U(N)}$. This justifies effective extension of the gauge group needed for diagonalizing Z. The rest of the computations are the standard Van der Monde determinant techniques [21] leading to

$$DZD\bar{Z}|_{1/2\ BPS} = \prod_{i>j=1}^{N} (z_i - z_j) \prod_{i>j=1}^{N} (\bar{z}_i - \bar{z}_j) \prod_{i=1}^{N} dz_i d\bar{z}_i\ .$$

The Lagrangian in the 1/2 BPS sector, for diagonalized Z simply reduces to $\sum_{i=1}^{N} \bar{z}_i z_i$. This proves the statements made in (3.7), (3.8).

As the first outcome of the above discussion, we note that the 1/2 BPS sector of $\mathcal{N} = 4$ SYM is equivalent to a Laughlin wavefunction with $\nu = 1$.[2] This observation has also been made in [6]. For $\nu = 1$ the wavefunction is antisymmetric with respect to exchange of any two z_i, z_j and hence the eigenvalues of Z are describing positions of N fermions in the lowest Landau level (LLL).

Next, we note the important property of the wavefunctions of the system of N particles in the LLL: the wavefunction can be written as

$$\psi_{LLL} = f(z_i)\ e^{-\sum_{i=1}^{N} \bar{z}_i z_i/2}\ ,$$

where f, regardless of the statistics of the underlying particles and its symmetry behavior under exchange of z_i's, is a *holomorphic* function of z's [14]. This holomorphicity is then directly related to the fact that the chiral primaries (1/2 BPS operators of $\mathcal{N} = 4$ SYM) are holomorphic in Z.

Finally, as reviewed in section 2.1, in the Landau problem the coordinates of the 2d particles become noncommutative [14] and in the LLL particles essentially become one dimensional. In other words the $(z_i, \bar{z}_i)$ plane becomes the phase space of the particles [14, 3] and hence (in proper units) $[z_i, \bar{z}_j] = \delta_{ij}$. As reviewed briefly in the introduction, this has become manifest in the LLM setup [5].

The emergence of an integer Hall system from the 1/2 BPS sector of SYM finds a simple interpretation within the geometric description of this sector via LLM. As reviewed in the introduction, the smoothness condition for these geometries allows for only two boundary conditions for the function z_0 on the noncommutative plane $(z_i, \bar{z}_i)$. In terms of LLM's color coding, this means that the minimal area accessible to the black regions is the same as the one for the white regions. Alternatively one can say that the absence of a minimal black spot corresponds to the presence of a minimal white spot. On the other hand, for a QHS with $\nu \equiv 1/k$ (with integer k), the minimal area accessible to a particle is k times as that for a hole. As a result, the density of particles can acquire $k+1$ different values or alternatively the absence of a particle is equivalent to the presence of k holes. The two pictures can thus be related to one another only if $k = 1$.

Therefore, $\mathcal{N} = 4$ SYM in the 1/2 BPS sector is describing the same physics as a quantum Hall system with $\nu = 1$ and with a specific edge state. In what follows we elaborate further on the connection and relation of the two systems.

[2]The Laughlin wavefunction for a quantum Hall system with filling fraction ν is [14]

$$\psi_{Laughlin} = \prod_{i>j=1}^{N} (z_i - z_j)^{\frac{1}{\nu}}\ e^{-\sum_{i=1}^{N} \bar{z}_i z_i/2}\ .$$

3.2 QH solutions Vs. SYM operators

In this section we explore the correspondence between the 1/2 BPS operators of $\mathcal{N} = 4$ SYM and the physical states of an integer QHS. Dealing with a $U(N)$ gauge theory with definite N calls for a QHS with a finite number of particles which is provided by the Polychronakos' construction [13].

Let us first review the physical states of a finite QHS with arbitrary ν (we will later focus on $\nu = 1$). The physical states of this system can be found by quantizing the corresponding matrix model (2.18) which results in the quantum Calogero model with the following Hamiltonian (we take $\omega = B = 1$) [19, 13, 20]

$$H = \frac{1}{2}\sum_{n=1}^{N}(p_n^2 + x_n^2) + \sum_{n\neq m}\frac{k(k-1)}{(x_n - x_m)^2}\ , \tag{3.9}$$

where $\nu = 1/k$ (note that for $k = 1$ the second sum vanishes). The eigenstates of this system are well known and are given by N positive integers $(f_1, f_2, ..., f_N)$ (known as quasinumbers) such that $f_i + 1 > f_{i+1} + k$. The ground state of the QHS, $|0\rangle_{QH,k}$, is given by $f_i = k(N - i)$ and its energy will be $\frac{1}{2}(kN^2 + N(1 - k))$. For future use we express the states in terms of their excitations above the ground state by the nonnegative integers $(r_1, r_2, \cdots, r_N)$ where $r_i = f_i - k(N - i)$ such that $r_1 \geq r_2 \geq \cdots \geq r_N \geq 0$. As explained in [13], these states describe N independent harmonic oscillators with an enhanced exclusion principle such that any two occupied states can not be closer than k. For $k = 1$, the particles will thus be ordinary fermions.

One can, on the other hand, find the states of $\mathcal{N} = 4$ SYM in the 1/2 BPS sector. In [3, 4], these states have been found by quantizing a one matrix model with a harmonic oscillator potential with unit frequency. As described in this paper, different gauge fixings for the model result in different bases for the spectrum. The first one, trace basis, consists of N^2 free bosonic harmonic oscillators subject to the constraint that the states must be neutral under gauge transformations. These states are expressed in terms of a set of positive integers $(c_1, c_2, \cdots)$ such that $N \geq c_1 \geq c_2 \geq \cdots$. Each number n in this set is attributed to a creation operator $\beta_n^\dagger$ which acts on the gauge invariant vacuum $|0\rangle_{tr}$. The ground state energy for the vacuum will thus be $N^2/2$. The upper bound on c_i comes from the fact that operators with $n > N$ are not independent.

A second gauge choice, eigenvalue basis, leads to a system of N free fermionic oscillators. The states are expressed in terms of N nonnegative integers $(f_1, f_2, ..., f_N)$ such that $f_1 > f_2 > \cdots > f_N$. The ground state, $|0\rangle_{EV}$, is given by $f_i = N - i$ and its energy will be $\sum_{n=0}^{N-1}(n + 1/2) = N^2/2$. It is more convenient to write the states in terms of nonnegative integers which represent the excitations above the ground state $(r_1, r_2, \cdots, r_N)$ where $r_i = f_i - (N - i)$ such that $r_1 \geq r_2 \geq \cdots \geq r_N \geq 0$. It was shown in [3] that the above two bases are related by the Schur polynomial basis and the two descriptions are identical. There is then a one-to-one correspondence between the fermionic occupation indices f_i's (or r_i's) and the bosonic harmonic

oscillator c_i's. The map between the two is basically given by the bosonization of the 2d fermion system. (This point has been discussed in a recent paper [22].) One may also use the difference of two successive r_i's for labeling the states, i.e. $(w_1, w_2, \cdots, w_N)$, where $w_i = r_{i-1} - r_i$ for $1 < i < N$ and $w_N = r_N$. In the Young tableau notation the w_i labeling corresponds to the Dynkin labels, see Fig. 1.

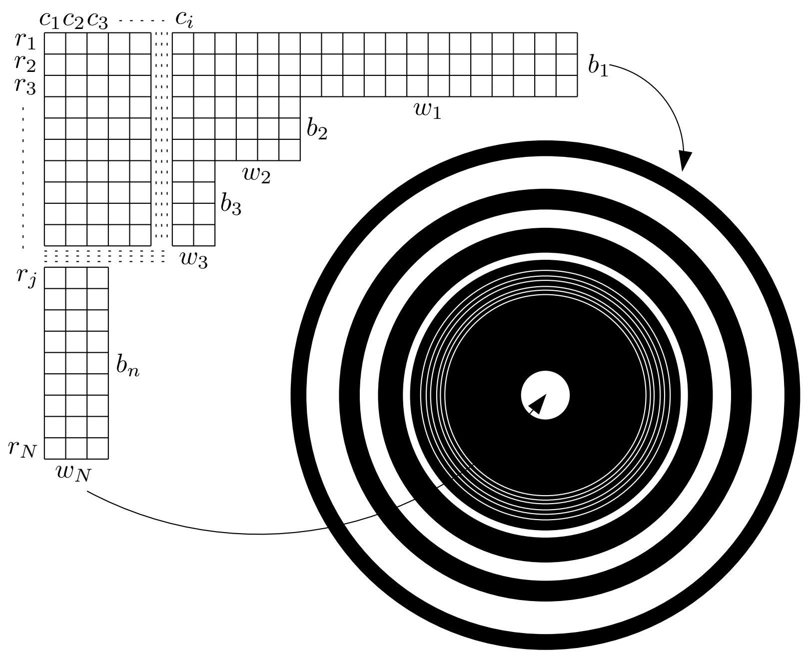

Figure 1: A generic Young tableau is in one-to-one correspondence with an LLM configuration of black and white circular rings, which is also equivalent to a similar configuration of quantum Hall droplets. There are various ways to label a Young tableau, each suitable for a different interpretation in the quantum Hall system. For example one can identify a tableau by the length of its rows (r_i's) or length of its columns (c_i's) or the Dynkin labels, $(b_1, w_1; b_2, w_2; b_3, w_3; ...; b_n, w_n)$ (b_i is counting the number of zeros in the standard Dynkin labels). The latter directly correspond to the area of black and white regions in the quantum Hall droplet picture. The r_i's and c_i's are related by bosonization of 2d fermion system.

We now have all the ingredients to relate finite QHS to the 1/2 BPS sector of the gauge theory. This relation is most apparent when we use the eigenvalue basis for the latter. In this basis, it is immediately seen that the two descriptions are identical if and only if $k = 1$. It is an interesting and curious question whether one can find deformations or perhaps other sectors of SYM which could be described by fractional QHS i.e. with $k \neq 1$. We will address this question in the discussion section.

Restricting to $k = 1$, consider for example the ground state $|0\rangle_{QH,1}$. This is a circular droplet with radius $R \sim \sqrt{N}$ and maps to the ground state of the gauge

theory or $|0\rangle_{EV} \equiv |0\rangle_{tr}$ which in turn corresponds to the $AdS_5 \times S^5$ background with $R^2_{AdS} \sim \sqrt{N}$. The minimal excitation of the particle system amounts to exciting the particle at the edge of the droplet to the state $|N\rangle$ which produces a circular hole state near the edge of the droplet. In the eigenvalue basis this is the state $(1, 0, ..., 0)$ and in the trace basis it is produced by $\beta_1^\dagger|0\rangle_{tr}$. The corresponding geometry is an $AdS_5 \times S^5$ with a smeared giant very close to the equator of S^5. As another example, a hole in the origin of the droplet, $(1, 1, ..., 1)$, is identical to $\beta_N^\dagger|0\rangle_{tr}$ and corresponds to the largest giant on the pole of S^5. A hole of unit area in a generic point in the quantum Hall droplet, which is described by a coherent state in the quantum Hall language, corresponds to a localized giant graviton the radius squared of which is proportional to its distance from the edge of the droplet.

4 Quasiholes in the $\mathcal{N} = 4$ SYM

In this section we attribute independent physical degrees of freedom to the hole states of QHS by considering them as the excitations of a dynamical field which we denote by Φ. In this way, the hole states are put on the same footing as the particles and the duality between these states is promoted to a dynamical one.

There are several motivations why one would like to treat the two different excitations (particles Vs. holes) symmetrically. A strong one comes from the gravity description of QHS in the LLL with unit filling fraction ($\nu = 1$) which is given by the half BPS geometries (LLM solutions) of type IIB SUGRA. (This point has first been discussed in [23].) In fact, for these geometries the above mentioned duality is enhanced to a symmetry which manifests itself through the invariance of solutions under the interchange of black and white boundary conditions, accompanied by a change of orientation on the plane (x_1, x_2). This enhancement occurs because for $\nu = 1$ the statistics of particles, given by ν, is the same as that for the holes, given by $1/\nu$, and both are fermions. Furthermore, the symmetry requires the same absolute value for the charge units of particles and holes which holds only when $\nu = 1$.

From the quantum Hall physics side, adding new degrees of freedom has been considered in order to accommodate quasiparticles as well as quasiholes. The latter, as we reviewed in section 2.4, can be achieved via the edge state (*cf.* (2.26)). Quasiparticle states, however, were obtained within the model introduced in [24] in the context of commutative Chern-Simons model and then in [25, 26] extended to the noncommutative Chern-Simons.

4.1 Particle-Hole Lagrangian

To construct a symmetric model we introduce a new dynamical field Φ, in addition to Z, in the NCCS matrix theory and interpret the excitations of this field as the hole states. We choose Φ as an infinite dimensional complex matrix which is in the adjoint of the gauge group. The important point is that although we are dealing with

an infinite four matrix theory, as we will see, one can find solutions to this theory which represent finite particle/hole systems. Therefore the gauge group dimension in the particle/hole sector of the system comes up as a part of the solution rather than as an input parameter.

Let us start with the NCCS matrix model Lagrangian we presented in section 2.3

$$L_{CS} = -\frac{\pi\kappa}{\theta} Tr\bigg(- \epsilon_{ij} X_i (\dot{X}_j + i[A_0, X_j]) + 2\theta A_0 \bigg), \tag{4.1}$$

where in terms of external magnetic B field applied to QH liquid in the unit of electric charge

$$2\pi\kappa = B\theta = \frac{1}{\nu} \ , \tag{4.2}$$

and ν is the filling fraction. If we expand the covariant position operator X_i around the solutions of the equation of motion for A_0, $[X_i, X_j] = i\theta\epsilon_{ij}$, in terms of the comoving coordinates y_i's as

$$X_i \equiv y_i + \theta\epsilon_{ij} A_j \ , \tag{4.3}$$

with $[y_i, y_j] = i\theta\epsilon_{ij}$, (4.1) then reduces to noncommutative Chern-Simons (NCCS) action in 2+1 dimensions [12, 25]. In this notation

$$\begin{aligned} D_i\Phi &\equiv \partial_i\Phi + i[A_i, \Phi] \\ &= \frac{i}{\theta}\epsilon_{ij}[X_j, \Phi] \,, \end{aligned} \tag{4.4}$$

where Φ is an arbitrary matrix which can also be thought of as a field in the adjoint representation of NC $U(1)$.

To incorporate holes as independent dynamical degrees of freedom we add a non-relativistic matter field Φ to the action (4.1):

$$\begin{aligned} L_{Z-\Phi} &= -\frac{\pi\kappa}{\theta} Tr\bigg(i(Z^\dagger D_0 Z - (D_0 Z)^\dagger Z) + 2\theta A_0 \bigg) \\ &+ \frac{\pi\kappa}{\theta} Tr\bigg(i(\Phi^\dagger D_0 \Phi - (D_0\Phi)^\dagger \Phi) - \frac{1}{2m} D_i\Phi (D_i\Phi)^\dagger - V(\Phi) \bigg), \end{aligned} \tag{4.5}$$

where $Z = \frac{1}{\sqrt{2}}(X_1 + iX_2)$ and $D_0 Z = \partial_0 Z + i[A_0, Z]$ (and similarly for Φ). m is the effective mass for the Φ particle and we choose the potential $V(\Phi)$ to be

$$V(\Phi) = -\frac{1}{2m\theta^2} \bigg([\Phi, \Phi^\dagger] + \frac{\theta}{2} \bigg)^2 . \tag{4.6}$$

The action (4.5) is an extension of the noncommutative version of Dunne-Jackiw-Pi-Trugenberger model [24], discussed in [26]. Note that, unlike [25], in our action Φ is in the adjoint (*cf.* (4.4)).

It is instructive to compare our Φ field with the edge state Ψ introduced in section 2.4. The Φ field being an $N \times N$ matrix, rather than an N vector, may be thought of as a collection of N number of edge states. In the classical analysis of the Polykronachos' model [13] with one edge state we could only describe a single droplet (which may have a hole in it *cf.* (2.26)). In order to describe two droplets, or generically multi-droplets, such as multi concentric rings in Fig 1, *classically* we need to introduce an edge state for each droplet. At quantum level, i.e. in Calogero model, quantum fluctuations of a single edge state, however, allows describing multi droplets. From the APD symmetry viewpoint this can be understood noting that APD's at classical level (w_∞ transformations) does not relate configurations with different number of rings while at quantum level ($W_{\infty+1}$ transformations) can tear the edge of a droplet apart and hence relate a single droplet to two droplets with the same area. In other words, the edge states corresponding to different number of rings belong to topologically distinct sectors of the gauge orbits of NC $U(1)$ in the NCCS theory or $U(N)$ in the Chern-Simons Matrix theory (4.1). In this viewpoint our model which *classically* contains the multi edge state field Φ, is an effective field theory description of the quantum version of the Polychronakos model (the Calogero model).

Before proceeding with the analysis of the action (4.5), let us motivate the potential term (4.6). As we have implicitly seen and would be discussed further in the following section, the Φ (and $\Phi^\dagger$) field corresponds to giant (anti-giant) gravitons in the gravity picture and $V(\Phi)$ is representing the giant-antigiant force and hence the potential (4.6) is a tachyonic potential corresponding to the open string tachyon in the giant-antigiant system. To first order in α' this potential can be obtained from the expansion of a Born-Infeld action. (Note that potential (4.6), up to integrals of total derivatives and a shift in zero point energy, is proportional to $-Tr([\Phi, \Phi^\dagger]^2)$.)

We start the analysis with the equation of motion for A_0, the Gauss law constraint

$$[Z, Z^\dagger] - [\Phi, \Phi^\dagger] = \theta \ . \tag{4.7}$$

Comparing (4.7) with (1.1), it is convenient to choose $\theta = 2\pi l_p^4$ and $m\theta = l_p$ and use the units in which Z and Φ are both measured in units of $\sqrt{\theta}$, ∂_0 and A_0 in units of $1/l_p$. As discussed earlier, in the 1/2 BPS sector of $\mathcal{N} = 4$ SYM we can only realize a quantum Hall system with $\nu = 1$ and hence we set $2\pi\kappa = 1$.

In the following we will show that this Lagrangain admits BPS (solitonic) solutions. In order to do that we begin with the Hamiltonian

$$H = \frac{1}{2\theta}\left[\frac{1}{2m\theta^2} Tr([Z, Z^\dagger][\Phi, \Phi^\dagger] - 2[Z^\dagger, \Phi^\dagger][Z, \Phi]) + TrV(\Phi)\right] . \tag{4.8}$$

Recalling the form of the potential (4.6), and using the Gauss law constraint (4.7), it is readily seen that

$$[Z, \Phi] = 0 \tag{4.9}$$

appears as the BPS condition. We should stress that to obtain a BPS configuration (4.9) should be solved together with (4.7).[3] From the equations of motion for Z, it is inferred that for the static BPS configurations

$$A_0 = \frac{1}{4m\theta^2}\left([\Phi,\Phi^\dagger] + \frac{\theta}{2}\right) = \frac{1}{4m\theta^2}\left([Z,Z^\dagger] - \frac{\theta}{2}\right) . \tag{4.10}$$

In the second equality we have used the Gauss law constraint (4.7). For the BPS configurations the Hamiltonian (4.8) becomes a constant and

$$[Z+\Phi^\dagger,(Z+\Phi^\dagger)^\dagger] = \theta \ . \tag{4.11}$$

The action evaluated on a BPS configuration is

$$L^{BPS}_{Z-\Phi} = -\frac{i}{2\theta}\left(Z^\dagger\partial_0 Z - (\partial_0 Z)^\dagger Z - \Phi^\dagger\partial_0\Phi + (\partial_0\Phi)^\dagger\Phi\right) .$$

Φ fields which solve the BPS equation (4.9), if we treat Φ as a function of Z and $Z^\dagger$, are all holomorphic functions of Z. This fact can directly be connected with the holomorphicity of the chiral primary operators in $\mathcal{N}=4$ SYM and/or the holomorphicity of the wavefunctions describing a quantum Hall system in the lowest Landau level (i.e. Laughlin wavefunction). In other words, the BPS configurations of the action we have proposed are satisfying the same condition as the (half) BPS configurations of the $\mathcal{N}=4$ SYM. It is worth noting that, as seen from (4.7), the classical solutions, BPS or non-BPS, to the Z-Φ model are all infinite size matrices.

4.2 Static solitonic BPS solutions

In this section we find some static classical solutions to the BPS equations derived from Lagrangian (4.5). The BPS configurations can be represented by the color coding: black region for the particles, where $[Z,Z^\dagger]$ is non-zero, and white region for quasiholes, where $[\Phi,\Phi^\dagger]$ is non-vanishing.

I. Vacuum

As the first solution we consider the vacuum. This can be chosen to be either the particle vacuum (black plane) or the hole one (white plane). Let's choose the latter for which the solution is simply given by

$$Z = 0\ , \qquad \Phi = \sum_{n=1}^{\infty}\sqrt{n\theta}|n\rangle\langle n-1|\ . \tag{4.12}$$

[3] As has been discussed in [26], BPS solitonic equations (4.7), (4.9) can be obtained from a noncommutative version of 2d chiral model. The reduction from (2 + 1) dim. to 2d may be understood noting that BPS solitons are time independent. In [26] it has also been argued that noncommutative 2d chiral model is solvable and the moduli space of the solutions is trivial.

This solution for Φ can be interpreted as an in finite number of concentric annular rings with unit area around the origin. The inner and outer radii of the ring n, $R_{i(n)}$ and $R_{o(n)}$, can be determined by

$$R_{o(n)}^{\ 2} = \theta \langle n|\Phi_{n+1}^\dagger \Phi_{n+1}|n\rangle \ \ , R_{i(n)}^{\ 2} = \theta \langle n|\Phi_n \Phi_n^\dagger |n\rangle \ , \tag{4.13}$$

where

$$\Phi_n \equiv \sqrt{n\theta}|n\rangle\langle n-1| \ , \qquad n = 1, 2, \cdots \tag{4.14}$$

such that $R_{o(n-1)} = R_{i(n)} = \sqrt{n\theta}$. The Gauss law constraint implies that

$$R_{o(n)}^{\ 2} = \theta + R_{i(n)}^{\ 2} \ . \tag{4.15}$$

II. Black circular droplet

A black circular droplet is specified by the following solution

$$\begin{aligned} Z &= \sqrt{\theta} \sum_{n=1}^{N} \sqrt{n}|n-1\rangle\langle n| \ , \\ \Phi &= \sqrt{\theta} \sum_{n=N+1}^{\infty} \sqrt{n}|n\rangle\langle n-1| . \end{aligned} \tag{4.16}$$

Here, the ring N has its inner radius inside the particles and its outer radius inside the holes. One can see that there is an anomaly in $[Z, Z^\dagger]$ in the $|N\rangle\langle N|$ component which is removed by $\Phi_{N+1}^\dagger \Phi_{N+1}$. Comparing this to the finite matrix model of Polychronakos, it is obvious that $\Phi_{N+1}^\dagger \Phi_{N+1}$ is behaving as $\Psi\Psi^\dagger$ where Ψ is the edge state. Therefore the edge state for the particle sector is provided by the hole degrees of freedom. Alternatively, an edge state for the hole sector is provided by the particle degrees of freedom through the term $Z_N^\dagger Z_N$. This will be discussed further in section 4.3.

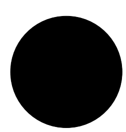

Figure 2: A circular droplet

III. White rings inside a black circular droplet

As the next example we present the solution corresponding to a black circular droplet with a concentric white ring inside as depicted in Fig. 3.

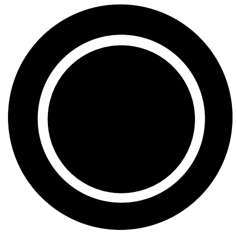

Figure 3: A circular droplet with a white ring. The area of the inner black disk is k_1, the white ring k_2 and the outer black ring k_3. This BPS solitonic configuration of our Z-Φ model corresponds to k_2 smeared S^5-giant gravitons in a $U(k_1+k_3)$ gauge theory or alternatively to k_3 smeared AdS-giants in a $U(k_1)$ gauge theory.

The solution is given by

$$
\begin{aligned}
Z &= \sqrt{\theta}\Bigg(\sum_{n=1}^{k_1}\sqrt{n}|n-1\rangle\langle n| + \sum_{n=q+1}^{q+k_3}\sqrt{n}|n-1\rangle\langle n|\Bigg) , \\
\Phi &= \sqrt{\theta}\Bigg(\sum_{n=k_1+1}^{q}\sqrt{n}|n\rangle\langle n-1| + \sum_{n=q+k_3+1}^{\infty}\sqrt{n}|n\rangle\langle n-1|\Bigg) ,
\end{aligned}
\tag{4.17}
$$

where $q = k_1 + k_2$ and k_1, k_2, k_3 are respectively areas of the inner black, white and outer black regions.

IV. Plane-wave solution

The plane-wave can be found either as an independent solution or as a limit of the droplet. This amounts to pulling the top left corner of the Z and Φ matrices in the droplet solution to infinity. It is readily seen that the resulting configuration, as expected, describes a Fermi sea. Taking such a limit of the droplet solution with white rings inside, will yield the ladder configuration. One should note that in the latter case, the limit is possible if the original solution describes very narrow black and white rings near the edge of the droplet. This means that the nonzero islands in Z and Φ matrices which describe the black and white stripes respectively, must be very small compared to the dimension of the top left sub matrix of Z which describes the droplet.

4.3 Closer connection to $N=4$ SYM and QH states

A 1/2 BPS operator of $\mathcal{N}=4$ $U(N)$ SYM is characterized by a couple of quantum numbers. The first one which is not usually mentioned is the size of matrices N. The second is R-charge J. With a given J and N, the number of traces (or number of subdeterminants) in the operator constitute the other quantum numbers. As

reviewed in section 3.2 and depicted in Fig.1 all this information can be summarized in a Young tableau or configuration of concentric rings. It is instructive to extract all these information from our Z and Φ matrices.

It is readily seen that the only non-zero entries of the matrix $Z^\dagger Z$ for a BPS configuration, in the harmonic oscillator basis we have employed, are diagonal ones, where we have black regions. Therefore, the number of non-zero elements of $Z^\dagger Z$ is N. In order to represent N as trace over a matrix, we construct a regularized matrix inversion for any given matrix A

$$A^{-1}_{reg} \equiv \frac{1}{2} \lim_{\epsilon \to 0} \left(\frac{1}{A + \epsilon \mathbf{1}} + \frac{1}{A - \epsilon \mathbf{1}} \right) . \tag{4.18}$$

This definition of A^{-1} reduces to the standard one when A is invertible and has no zero eigenvalues. However, when A has zero eigenvalues is takes out the part with zero eigenvalues and inverts the rest of the matrix, e.g.

$$A = diag(a_1, a_2, \cdots, a_k, 0, 0 \cdots) \ \Rightarrow \ \ A^{-1}_{reg} = diag(\frac{1}{a_1}, \frac{1}{a_2}, \cdots, \frac{1}{a_k}, 0, 0 \cdots) \ .$$

Using (4.18) it is then easy to construct N as

$$N_b = Tr\left((Z^\dagger Z)(Z^\dagger Z)^{-1}_{reg} \right) \ . \tag{4.19}$$

θN_b is the area of the black region. Similarly one can define $N_w = Tr\left((\Phi\Phi^\dagger)(\Phi\Phi^\dagger)^{-1}_{reg} \right)$ as the area of the white region. In our case we have chosen N_b to be finite while N_w is infinite. Obviously both N_b and N_w are conserved quantum numbers on the space of BPS solutions, as they are all static.

The quantum number J is associated with the rotation symmetry in the Z-plane. In other words, J is the Neother charge for the rotations $Z \to e^{i\phi} Z$, i.e.

$$J = Tr(Z^\dagger Z) - \frac{1}{2} N_b (N_b + 1) \ , \tag{4.20}$$

where we have subtracted off the "zero point energy" $\frac{1}{2} N_b (N_b + 1)$ to match with the conventions of quantum Hall system and the super Yang-Mills. One can also define $J_w = Tr(\Phi\Phi^\dagger)$, which is again the Neother charge associated with the rotations in the Φ-plane. It is easy to see that for the disk configuration (4.16) $N_b = N$ and $J = 0$ and for the ring solution (4.17) $N_b = k_1 + k_3$ and $J = k_2 k_3$.

In section 2.2 we discussed that the density of the quantum Hall fluid (in the Euler description) is $(\theta \epsilon_{ij} \{X^i, X^j\})^{-1}$ (2.10). Hence the inverse of $[Z, Z^\dagger]$ (or $-[\Phi, \Phi^\dagger]$) should correspond to the particle (hole) density

$$\rho_{part} = [Z, Z^\dagger]^{-1}_{reg} \quad , \quad \rho_{hole} = [\Phi^\dagger, \Phi]^{-1}_{reg} \ . \tag{4.21}$$

It is straightforward to see that for our concentric ring solutions in the harmonic oscillator basis $[Z, Z^\dagger]$ is diagonal and except for the boundaries of the black and

white regions, where we have color-changing, ρ takes values zero or one. This is as we expected and is made explicit in the LLM construction. For example for the single ring configuration given in (4.17)

$$\begin{aligned}\rho_{part} =& diag(\overbrace{1,1,\cdots,1}^{k_1},\frac{-1}{k_1},\overbrace{0,0\cdots,0}^{k_2-1},\frac{1}{k_1+k_2+1},\overbrace{1,\cdots,1}^{k_3-1},\frac{-1}{k_1+k_2+k_3},0,\cdots)\ ,\\ \rho_{hole} =& diag(\underbrace{0,0,\cdots,0}_{k_1},\frac{1}{k_1+1},\underbrace{1,1\cdots,1}_{k_2-1},\frac{-1}{k_1+k_2},\underbrace{0,\cdots,0}_{k_3-1},\frac{1}{k_1+k_2+k_3+1},1,\cdots)\ .\end{aligned} \tag{4.22}$$

Number of rings is then related to the number of non-integer elements of ρ. Alternatively, one can show that

$$r \equiv \sharp\text{Rings} = \frac{1}{2}Tr\left((\rho_{part}[Z,Z^\dagger]) - N_b - 1\right)\ . \tag{4.23}$$

As we see all the information of the Young tableau can easily be extracted from $Z^\dagger Z$ and $[Z,Z^\dagger]$ by methods similar to those mentioned above.

As discussed earlier, Φ matrices behave as (infinite) collection of Polychronakos edge states Ψ. To make a closer connection to the edge states, we note that diagonal elements of $[\Phi^\dagger,\Phi]$ (or $[Z,Z^\dagger]$) are either zero or one except on the edge of the rings. This happens in $2r+1$ points. Hence $[\Phi,\Phi^\dagger]^2+[\Phi,\Phi^\dagger]$ is zero except on the locus where we have color-changing.[4] Explicitly

$$[\Phi,\Phi^\dagger]^2+[\Phi,\Phi^\dagger] = \sum_{n=1}^{2r+1}\Psi_n\Psi_n^\dagger(\Psi_n\Psi_n^\dagger-1)\ , \tag{4.24}$$

where Ψ_n are the effective edge states we need to include in the NC Chern-Simons Matrix theory to describe a configuration with r rings.[5]

We should, however, emphasize an important difference between our Φ field and the edge state(s) Ψ. As mentioned earlier, our Φ field is a classical (effective) field theory description of the quantized Polychronakos model. In the droplet with a hole solution (2.26) the size of the hole q is not quantized. In our model, as it can be seen from (4.17) (by setting $k_1=0$) the area of the hole is quantized. In our Z-Φ symmetric model, unlike the Polychronakos model,[6] the area of the white and black regions, are *quantized* both in the *same* units, θ. The latter is a property of quantum

[4] A similar observation has been made in [30]. There, however, it was proposed to take $[\Phi,\Phi^\dagger]^2+[\Phi,\Phi^\dagger]=0$, ignoring the edge effects.

[5] We would like to comment that in the Polychronakos terminology in fact we do not need $2r+1$ edge states and $r+1$ of them is enough; whenever we are moving from a black region to white region we need an edge state. Or in terms of our matrices that is the number of negative eigenvalues of the $[Z,Z^\dagger]$ matrix.

[6] It is notable that in the quantized Polychronakos model, the Calogero model, the area of the holes q is also quantized [13].

Hall systems with $\nu = 1$. As the other manifestation of this "quantization" we note that in our model the transition between black and white regions does not occur suddenly. For example, as it is seen from (4.22) there are $2r + 1$ points where the change of color happens. These are the places where ρ_{part} and ρ_{hole} have overlap.

4.4 More on symmetries of the Z-Φ model

As already discussed in the $\mathcal{N} = 4$ SYM a Young tableau may be interpreted as a configuration of sphere giants or AdS (dual) giants. On the other hand in principle, similar to the usual D-brane case, we have the option of having anti-giants of either kind. Of course in a 1/2 BPS sector with a given supersymmetry we only see giants or anti-giants and not both. There are two $\mathbb{Z}_2$ symmetries relating sphere giants and the AdS giants and/or giants and anti-giants [27]. In our Z-Φ model, in which giants and dual giants, respectively denoted by Φ and Z, appear as independent degrees of freedom one can realize both of the above $\mathbb{Z}_2$ symmetries. Both of these symmetries are essentially exchanging the black and white regions.

The BPS equation (4.9) is manifestly invariant under the exchange of Z and Φ and the Gauss law constraint (4.7), as well as (4.10), would remain unchanged if we also send $\theta \to -\theta$. In other words,

$$Z \longleftrightarrow \Phi \ , \tag{4.25a}$$

$$\theta \longleftrightarrow -\theta \ , \tag{4.25b}$$

is a symmetry of BPS configurations. (4.25b) in the LLM language means that besides the changing the black and white regions one should also change the orientation of the (x_1, x_2) plane (*cf.* (1.1)) [27]. The value of A_0 (4.10), noting the Gauss law constraint, remains unchanged under the above Z_2 symmetry. In the quantum Hall language the above is a particle$\leftrightarrow$quasihole symmetry.

The BPS configurations are invariant under another $\mathbb{Z}_2$ transformation which exchanges a giant with a dual anti-giant, i.e.

$$Z \longleftrightarrow \Phi^\dagger \ , \tag{4.26a}$$

$$t, A_0 \longleftrightarrow -t, -A_0 \ . \tag{4.26b}$$

In the above transformation we exchange black and white regions without changing the (x_1, x_2) plane orientation. In the quantum Hall terminology (4.26) which contains an inversion in time is a particle/anti-quasihole exchange symmetry. Although our BPS equations are $\mathbb{Z}_2$ invariant, their solutions are not necessarily symmetric, as it is manifest in the black-white diagrams. The only $\mathbb{Z}_2$ symmetric solution is when half of the plane is filled by the black region, corresponding to the plane-wave solution in the supergravity setup [5].

Compared to $\mathcal{N} = 4$ SYM the rank of the gauge group N, in our model appears as a characteristic of the specific solutions rather than a parameter in the Z-Φ Lagrangian. This observation opens the way for extensions of AdS/CFT in which

certain quantities of $U(N)$ and $U(M)$ gauge theories are related. In fact one can distinguish two different such extensions. The first is inspired by Fig. 3 according which there is a relation between giants of a $U(k_1+k_3)$ gauge theory and dual giants of a $U(k_1)$ gauge theory. This is a very remarkable result, if it can be extended to beyond the 1/2 BPS sector and to the full theory, as one can then always use a "dual" picture in which the rank of the gauge group is large and one can perform the 't Hooft planar-nonplanar expansion.

The other such duality between gauge theories of different rank may come from the above mentioned black/white exchange symmetry. Consider the static solutions of our particle-hole symmetric matrix model which describe a black droplet with circular white rings inside. (4.25) exchanges the giants and dual giants, if the Z's are scalars of a $U(N+M)$ gauge theory, one then expects that Φ's should become the scalars of a $U(M)$ theory and vice-versa. To make the argument more tractable, we take the matrices finite dimensional of size $N+M$ such that for each solution, $[Z, Z^\dagger]$ has N and $[\Phi, \Phi^\dagger]$ has M nonzero diagonal components.[7] Furthermore, we assume that $N \ll M$. The ground state of such a system will thus be described by a white 2-sphere with a very small black spot, on the north pole say. This state is given by

$$\begin{aligned} [Z, Z^\dagger]_{ij} &= \delta_{ij}\ , \quad i,j=1,2,\cdots,N\ , \quad [Z,Z^\dagger]_{N+1N+1} = -N\ , \\ [\Phi, \Phi^\dagger]_{ij} &= \delta_{ij}\ , \quad i,j=N+2,\cdots,N+M\ , \quad [\Phi,\Phi^\dagger]_{N+1N+1} = N+1\ . \end{aligned} \tag{4.27}$$

Excitations of the system are represented by nonzero diagonal islands in the commutators. Suppose an excitation which has an island of length n in $[Z, Z^\dagger]$ and an island of length m in $[\Phi, \Phi^\dagger]$. One can view this, as a 1/2 BPS excitation of a $U(N)$ $\mathcal{N}=4$ SYM produced by $({\beta_n^{(P)}}^\dagger)^m$ acting on the gauge invariant vacuum or, equivalently, as such an excitation of a $U(M)$ $\mathcal{N}=4$ SYM which is now produced by $({\beta_m^{(H)}}^\dagger)^n$ (where the superscript P (H) on β denotes Particle (Hole).) Remember that, as stated in 3.2, ${\beta_n}^\dagger$ is the operator that produces gauge invariant states in the trace basis by acting upon $|0\rangle_{tr}$ [3, 29].

In other words, the excitations of the black Fermi level can be related to the hole states by going from trace to sub-determinant basis in the $U(N)$ theory. On the other hand, the same particle states can be related to the excitations of the white Fermi level by going from sub-determinant to trace basis in the $U(M)$ theory. As a result, the trace (sub-determinant) operators of the two theories can be easily mapped to one another using the Z and Φ matrices.

[7] Of course in our Z-Φ model we are always dealing with infinite size matrices. In order to keep both N and M finite, in the LLM terminology, we should consider the case with compact (x_1, x_2) plane. This will bring further complications [5] which would be addressed in a future work [28].

4.5 Stability analysis of the BPS configurations

So far we have stated the BPS equations of our Z-Φ model, constructed and analyzed concentric ring solutions. In this section we study classical stability of these BPS configurations. To examine the stability let us start with the equations of motion for $Z^\dagger$ and $\Phi^\dagger$:

$$\begin{aligned} &D_0 Z - \frac{i}{4m\theta^2}[[\Phi, \Phi^\dagger], Z] + \frac{i}{2m\theta^2}[\Phi^\dagger, [Z, \Phi]] = 0\,, \\ &D_0 \Phi + \frac{i}{4m\theta^2}[[Z, Z^\dagger], \Phi] + \frac{i}{2m\theta^2}[Z^\dagger, [Z, \Phi]] + \frac{i}{2}\frac{\partial TrV}{\partial \Phi^\dagger} = 0\,. \end{aligned} \tag{4.28}$$

(The equations of motion for Z and Φ are similar to these equations obtained by getting a hermitian conjugate.) The above should be solved together with the equation of motion for A_0, the Gauss law constraint (4.7). In the gauge where A_0 is given by (4.10), the equations of motion simplify significantly to take the form

$$\dot{\Phi} + \frac{i}{2m\theta^2}[Z^\dagger, [Z, \Phi]] = 0\,, \tag{4.29a}$$

$$\dot{Z} + \frac{i}{2m\theta^2}[\Phi^\dagger, [Z, \Phi]] = 0\,, \tag{4.29b}$$

where dot denotes the time derivative.

To address the classical stability of the BPS configurations first we note that equations (4.29) are first order in time and hence there is no non-trivial solution which at $t = 0$ is a BPS configuration $\Phi(t = 0) = \Phi_0$ and $Z(t = 0) = Z_0$, where $[Z_0, \Phi_0] = 0$. Next let us perturb the BPS solution as

$$\Phi = \Phi_0 + \delta\Phi(t)\ , \quad Z = Z_0 + \delta Z(t) \tag{4.30}$$

and let us suppose that there is no time that $\delta Z(t)$ and $\delta\Phi(t)$ vanish simultaneously (otherwise the perturbations would vanish for all t). Plugging the above into (4.29) and (4.7) expanding up to the first order in perturbations we obtain

$$\begin{aligned} -i\frac{d}{d\tau}\delta Z + [\Phi_0^\dagger, [Z_0, \delta\Phi]] + [\Phi_0^\dagger, [\delta Z, \Phi_0]] = 0\ , \\ -i\frac{d}{d\tau}\delta\Phi\ + [Z_0^\dagger, [Z_0, \delta\Phi]] + [Z_0^\dagger, [\delta Z, \Phi_0]] = 0, \\ [Z_0^\dagger, \delta Z] + [Z_0, (\delta Z)^\dagger] = [\Phi_0^\dagger, \delta\Phi] + [\Phi_0, (\delta\Phi)^\dagger]\ , \end{aligned} \tag{4.31}$$

where $\tau = 2m\theta^2 t$. In order to show the stability we should argue that the perturbations do not grow in time. This can be argued for noting the constants of motion. It can be directly checked that any quantity of the form $Tr(AA_{reg}^{-1})$, which is always integer valued, is (classically) conserved. (To see this it is enough to note (4.18).) In particular the area of the black region, N, and its second moment, J, number of

rings r and the Hamiltonian are conserved. Hence the perturbations cannot grow in time.

As an example we solve (4.31) for perturbation about the vacuum (the whole white solution) given through (4.12). For this background $Z_0 = 0$ and hence $\delta\Phi = 0$

$$\frac{d}{d\tau}\delta Z - i[\Phi_0^\dagger, [\Phi_0, \delta Z]] = 0 \ , \tag{4.32}$$

where $[\Phi_0, \Phi_0^\dagger] = -\theta$. It is readily seen that

$$\delta Z = e^{-i\omega\tau} e^{k\Phi_0} \ e^{p\Phi_0^\dagger} \tag{4.33}$$

solves (4.12) with $\omega = \theta^2 kp$. Assuming that δZ is finite for large distances on the x_1, x_2 plane we have $k = -\bar{p}$ leading to $\omega = -\theta^2|k|^2$. Since ω is real valued the perturbations does not grow in time. The (4.33), as expected is a non-relativistic wave of particles (black region) moving in e.g. x_1 direction. In general we expect that a generic solution to (4.31) to be a linear combination of the plane-waves of the form (4.33). Alternatively one may construct a solution to two dimensional wave equations with rotational symmetry, via the Bessel functions. The latter would be more appropriate for studying the fluctuations about the circular symmetric ring solutions we considered in section 4.2.

Finally, from the above discussions it is inferred that classically there is no transition between the BPS configurations. That is, there is no solution which at two times t_0 and t_1, $Z(t_0) \neq Z(t_1)$, is a BPS configuration. As discussed in previous section N and J are both conserved charges and if there is any transition, classically or quantum mechanically, between various BPS solutions the initial and final states should have the same J and N. A quantum mechanical transition between the rings, however, imply that the conservation of number of rings r, which holds classically, is violated quantum mechanically. In fact it is not difficult to find paths (of course not classical ones) which interpolate between the BPS configurations which are not the dominant contributions compared to the full $\mathcal{N} = 4$ SYM theory analysis [4, 9].

5 Discussion and Outlook

In this paper we have studied, refined and clarified the correspondence between the quantum Hall system and the $\mathcal{N} = 4$ SYM in the 1/2 BPS sector and the LLM geometries, which was discussed in [5, 6]. In this viewpoint the 1/2 BPS sector of $\mathcal{N} = 4$ SYM is equivalent to a QHS with filling fraction $\nu = 1$. We showed the equivalence of the two at the level of the actions and the Hilbert spaces and/or the partition function of the $\mathcal{N} = 4$ SYM in the 1/2 BPS sector. We have shown that the (square of) the Laughlin wavefunction with $\nu = 1$ is nothing but the partition function of the $\mathcal{N} = 4$ SYM in the 1/2 BPS sector. In sum, we discussed and related four corners of the square depicted in the Fig. 4.

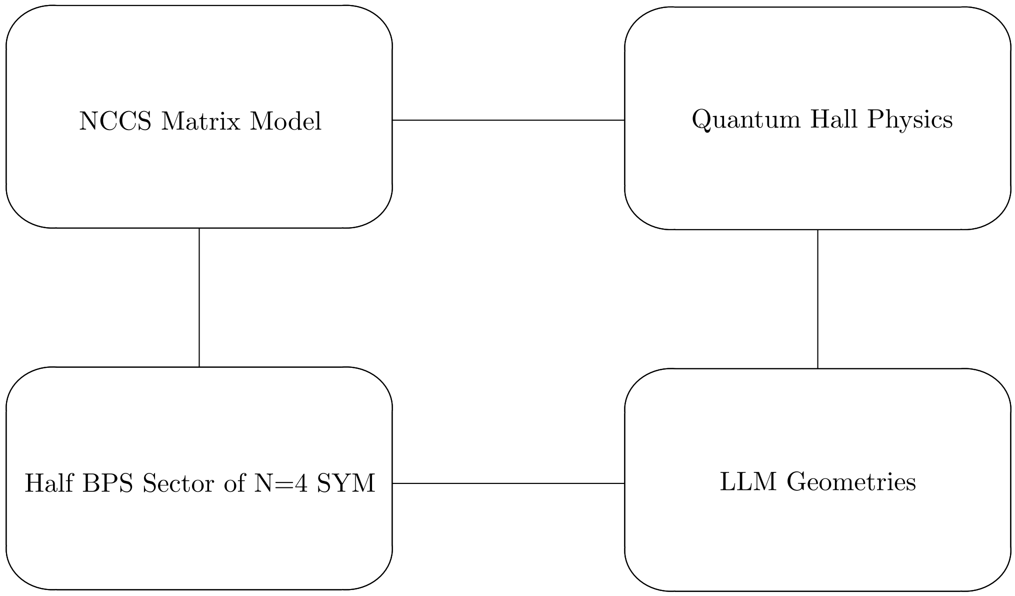

Figure 4: In this paper we have worked out and discussed relation of all the four corners of the above figure. The only case that we did not address here was a direct connection of LLM geometries and noncommutative Chern-Simons Matrix model. This may be done along the lines of minisuperspace quantization method for the LLM geometries. This point has been recently discussed in [32].

In the quantum Hall side the same machinery can be used for $\nu < 1$ systems, the fractional quantum Hall effect ($\nu > 1$ has problems with unitarity). We can now turn to the interesting question of the relevance of FQHS in the context of AdS/CFT, an issue which we advertised for in the paper. As mentioned before, the physical states of a finite FQHS with $\nu = 1/k$ can be found as the states of a quantum Calogero model. The particles of this system, obeying an enhanced Pauli exclusion principle, occupy the energy states of a harmonic oscillator such that no two states closer than k are occupied. This results in the fact that on the noncommutative plane where the particles live, the area quantum for the particles is k times bigger than that for the holes but since we are probing the plane by the particles, and not with the holes, $k + 1$ possible values are found for the particle density. Therefore one can take an equivalent but more convenient picture of the system; think of particles as ordinary fermions but of holes as obeying a "reduced" exclusion principle such that at most k number of these objects can be put on top of one another. Such a "reduction" cannot happen for particles which is a result of the upper bound on $\nu(\leq 1)$ and thus accumulation of particles is not allowed.

One can make an analogous argument for the LLM geometries. These solutions have two important features; noncommutativity of (z_1, z_2) and smoothness condition $(z_0 = \pm 1/2)$. The first one arises upon the identification of the boundary plane with

the phase space of the dual fermionic system. Therefore, this feature is originating from the quantum nature of the corresponding QHS. In the geometric language, this noncommutativity is translated through the statement that even a single giant graviton in the AdS background produces a gravitational back reaction such that the boundary configuration for the resulting geometry is depicted as a white spot of minimal area inside a black droplet. This noncommutativity of the (x_1, x_2) plane is a result of quantum gravity considerations, coming from the dual SYM picture, via AdS/CFT. The smoothness condition of LLM geometries, on the other hand, is a statement about the statistics of the particles and holes in the dual QHS. As was found in [31], in order to exclude geometries with Closed Time-like Curves (CTC), one has to impose an upper bound on the boundary value $z_0 \leq 1$. In parallel lines with the above arguments one can say that the dual giants by which one probes the LLM backgrounds cannot be put on top of one another, a "stringy exclusion principle" [7], which is a result of the exclusion of CTC's in the geometries ($z_0 \leq 1$). On the other hand, giant gravitons also obey a "stringy exclusion principle" which is not reduced because the regularity of solutions does not allow for $-1/2 < z_0 < 1/2$ and hence *there are no coincident giants or dual giants in the LLM setup.* Namely, LLM do not have such solutions, because they only work with black and white regions. In their setup this is a condition coming form smoothness of the solutions. The latter have been made manifest in our Z-Φ symmetric model where Φ is associated with giants and Z with dual giants.

In other words, Pauli exclusion principle which is there for both quasiholes and particles in a quantum Hall system with $\nu = 1$, is a manifestation of the "stringy exclusion principle" coming out of giants and dual giants considerations. (The stringy exlcusion principle in the context of supergravity has also been discussed in [23].)

As stated here the $\mathcal{N} = 4$ SYM in the 1/2 BPS sector is related to quantum Hall system with $\nu = 1$. One interesting open question is whether the QHE/SYM correspondence can be pushed to $\nu < 1$ where we have the possibility of fractional quantum Hall effect and anyons. From our discussions it is inferred that such correspondence, if it exists, cannot happen on the 1/2 BPS sector and we need to go beyond this sector. One possibility, which is a direct outcome of the discussions of the above paragraphs about the fractional statistics of the quasiholes in the $\nu < 1$ cases, is to consider orbifolding in the $Z, Z^\dagger$ or $\Phi, \Phi^\dagger$ plane. That is, considering a S^5/Z_k orbifold of the $AdS_5 \times S^5$ geometry where the orbifolding is keeping an $SO(4)$ and is only acting on the S^1 in the $Z, Z^\dagger$ plane. This orbifold has a fixed point at $Z = 0$ and is breaking all the supersymmetry. Upon orbifolding we are identifying k slices of the black disk, that is as if we have k particles of fractional statistics on top of each other. This can happen if $\nu = 1/k$. This proposal immediately tells why ν^{-1} should be quantized. On the other hand, one can show that the superstar solutions [33], which are 1/2 BPS solutions of IIB supergravity with a naked singularity, are effectively behaving like a quantum Hall system with $\nu < 1$ (more precisely, $\nu = \frac{1}{1+q}$ where q is an integer related to the R-charge of the solution.) The above proposal then suggests that there should be a relation between the orbifold singularity and

the naked singularity of the superstar. Further exploration of different aspects of this idea is postponed to future works.

In the 1/2 BPS sector one can compute transition amplitudes between giant gravitons. That is basically the computation of three point function of three chiral primary operators carried out in [4]. This possibility has been ignored in the LLM setup. In our Z-Φ model, as we argued our BPS solutions are classically stable and as a result the number of giants is a conserved quantity. (In the Calogero model there is no possibility of such transitions between ring solutions, as states with different number of rings (giants) are orthogonal eigenstates of the Calogero Hamiltonian.) In the Z-Φ model, however, one has the possibility of quantum tunneling between giants and hence number of giants is not conserved once the quantum (non-perturbative instanton) effects are considered. This expectation is compatible with the picture advocated in section 2.4 of [34].

As another interesting extension one may try to push the QHE/SYM correspondence to beyond the 1/2 BPS sector. For example let us consider a part of the 1/4 BPS sector only containing operator made out of two of the three complex scalars, say Z and Y. As we argued in section 3.1 one may think of the angular momentum (R-charge) operator J as the effective action for the BPS sector, roughly that is $Tr(Z\frac{\delta}{\delta Z}) + Tr(Y\frac{\delta}{\delta Y})$. This is very similar to a four dimensional quantum Hall system. This is an open direction in need of further analysis.

The last interesting open question we would like to briefly discuss is the "duality" we alluded to at the end of section 4.4. To argue for the "duality" between $U(N)$ and $U(M+N)$ gauge theories we made the assumption that the total area of the black and white regions is finite. In the LLM terminology, that is the (x_1, x_2) plane is compact and since (x_1, x_2) plane is flat, our choices are limited to tori. (Note the footnote at the end of page 7 in [5] for a comment on this issue.) Since (x_1, x_2) plane is noncommutative, this torus should be a fuzzy torus. Exploring the possibility of compactifying (x_1, x_2) plane from gravity and/or QH system side and its implication of that for the "duality" mentioned above is left for future works.

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